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1989 J. Phys. A: Math. Gen. 22 L633

(http://iopscience.iop.org/0305-4470/22/13/018)

J. Phys. A: Math. Gen. 22 (1989) L633-L639. Printed in the UK

LETTER TO THE EDITOR

KAM tori, chaotic motion and exactly integrable islands for a non-differentiable Hamiltonian system

W Kob and R Schilling

Institute of Physics, University of Basel, CH-4056 Basel, Switzerland

Received 10 November 1988, in final form 1 February 1989

Abstract. A simple Hamiltonian system with two degrees of freedom is investigated numerically. Although the Hamiltonian is not everywhere differentiable but continuous, we nevertheless find KAM tori, showing that not all the assumptions of the KAM theorem are necessary. Furthermore there exist islands in phase space, embedded in a chaotic region, where the system seems to be exactly integrable.

As is well known, the bounded orbits of an *integrable* Hamiltonian system lie on tori filling the phase space. The survival of these tori under a *non-integrable* perturbation is a question of great theoretical and practical importance. Since Kolmogorov conjectured that part of the tori survive if the perturbation is analytic, and this was proved by Arnold, the assumptions for their survival have become weaker with time. For instance, Moser [1] has shown that it is sufficient to assume the perturbation to be M times continuously differentiable with $M \geqslant 4$, but he conjectured that $M=3$ is sufficient. Numerical examples have been found where invariant tori exist with $M=2$ [2, 3] and others where no tori exist for $M=2$ [4]. Thus there have been speculations [2] that $M=3$ is always sufficient and $M=2$ is necessary for the theorem to hold. But some years ago these speculations were shown be be wrong by constructing dynamical systems with $M=0$ still possessing invariant tori [5-8]. The slight shortcoming of the systems discussed in [6, 7] is their artificiality from a physical point of view.

In this letter we introduce a simple Hamiltonian system with two degrees of freedom. Although the Hamiltonian is not everywhere differentiable, we have strong numerical evidence for the existence of KAM tori, thus giving another, more physical, example for the presence of tori in the $M=0$ case. A further interesting property of this model is the apparent coexistence (on the same energy hypersurface!) of chaotic regions and of exactly integrable islands, i.e. regions in which no stochastic behaviour seems to exist. This feature has already been observed in other systems [7, 9, 10] but these were constructed such that this feature occurs.

The model we investigate is a two-particle version of a one-dimensional Hamiltonian used to describe disordered materials [11] and has the form

$$H=\frac{1}{2m}(p_1^2+p_2^2)+\frac{C_1}{2}[(q_1-a_+)^2+(q_2-a_+)^2]+\frac{C_2}{2}(q_1+q_2-b)^2$$
$$-C_1a_-[(q_1-c)\,\mathrm{sgn}(q_1-c)+(q_2-c)\,\mathrm{sgn}(q_2-c)]-C_1(c-a_+)^2 \qquad (1)$$

0305-4470/89/130633+07$02.50

i.e. it consists of two harmonically coupled particles, each of which feels an external, anharmonic potential given by patching together two identical parabolae at a given point c (see curve a in figure 1). This Hamiltonian has several important properties.

(i) It is not differentiable at $\{(q_1, q_2)|q_1 = c \text{ or } q_2 = c\}$.

(ii) The configuration space is divided into four disjoint sectors given by $\{(q_1, q_2)|q_1 > c, q_2 > c\}$, $\{(q_1, q_2)|q_1 < c, q_2 > c\}$, etc. Inside each sector H is harmonic and the system feels the anharmonicity only if one of the particles crosses the patching point at c.

(iii) The potential energy is symmetric with respect to interchange of q_1 and q_2.

The piecewise linearity of the corresponding equation of motion makes the numerical integration of a trajectory unusually easy. As long as the system stays in one of the sectors, the solution can be obtained analytically. Given a starting point in phase space, one has to calculate only the first time at which the system leaves the sector in which it started, i.e. one seeks for the smallest roots t_1, t_2 of the transcendental equations

$$q_1(t_1) = c \qquad q_2(t_2) = c. \tag{2}$$

Here $q_1(t)$ and $q_2(t)$ are analytically given expressions depending on the initial point. Having obtained $t^* = \min\{t_1, t_2\}$, one takes the state of the system at time t^* as the new initial point, but now for the new sector. The calculation of points for Poincaré sections can be done in a similar way. Note that the only numerical errors introduced in the calculations stem from finding the roots of the transcendental equations and are certainly much smaller than an error one would obtain by integrating the equations of motion numerically with a standard algorithm for solving non-linear differential equations. This fact allows the integration of the trajectories to a much higher precision

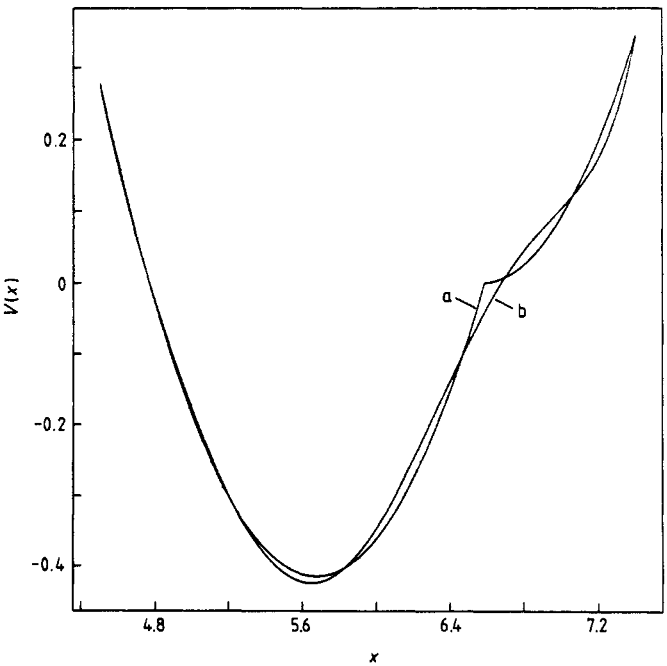

Figure 1. Curve a represents the external potential in Hamiltonian (1). Note the cusp destroying the global differentiability. Curve b shows a smooth approximation of the potential a.

compared to other dynamical systems (for the same amount of computer time). Typical integration times were 10^4-10^5 time units with typical timescale of the system of order 2π. This results in typically 10^3-10^4 Poincaré points.

For all calculations we have chosen the following values for the parameters:

$C_1 = C_2 = -0.217\,398\,372\,368\,26$ $\quad a_+ = 6.123\,4252$ $\quad a_- = 0.447\,648\,4686$

$c = 6.585\,757\,438\,67$ $\quad b = 10.743\,688\,383\,8474.$

In figure 2 we present Poincaré sections of the system with the surface given by $q_2 - 7 = 0$ and $p_2 \geq 0$ for different energies E. Several features can be observed.

(i) For E smaller than a critical value E_c (which is the energy of the lowest saddle point of the potential energy in configuration space) the particle cannot escape a harmonic sector. Thus for $E \leq E_c$ the system is integrable. In our case $E_c \simeq -0.809$.

(ii) For all energies larger than E_c there exist invariant tori as well as chaotic orbits with finite measure (see figure 2). This behaviour, both regular and irregular, was confirmed by the calculation of the corresponding Lyapunov exponents.

(iii) The measure of the chaotic region increases (for $E \leq 0$) with energy and takes a maximum at $E \simeq 0$ (cf figures 2(*a*) and 2(*c*)). A further increase of E leads to a decrease of this measure (cf figure 2(*d*).

(iv) For $E \to \infty$ the system becomes integrable again. This behaviour, which has been already observed in other dynamical systems [12], can easily be explained as follows. Scaling the coordinates q_i and the momenta p_i $(i = 1, 2)$ with $\sqrt{E}$, i.e., introducing $x_i = q_i/\sqrt{E}$ and $y_i = p_i/\sqrt{E}$, one obtains an equation of motion with a non-linearity proportional to $1/\sqrt{E}$. Therefore the limit $E \to \infty$ leads to integrability. Figure 2(*d*), where two large regions of regular behaviour can be identified, demon-

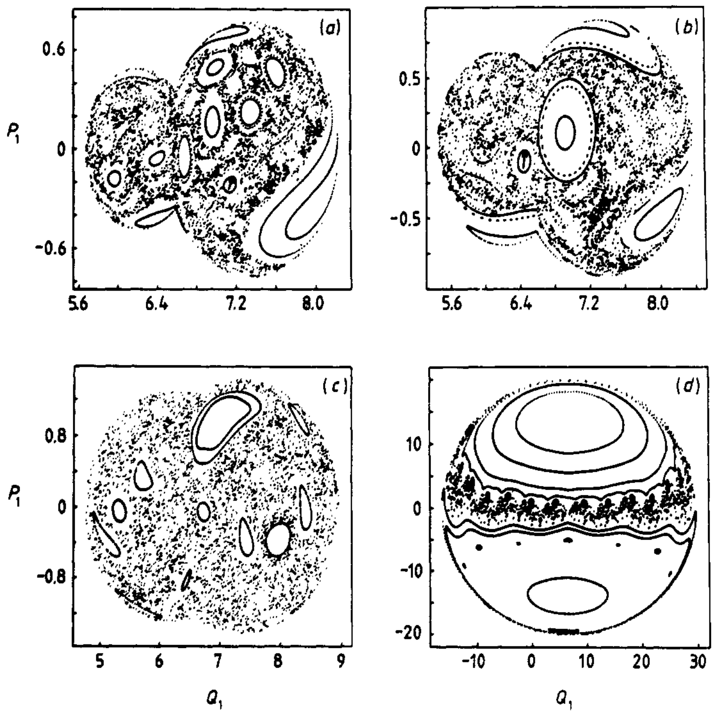

Figure 2. Poincaré sections for the system at different energies: (*a*) $E = -0.72$, (*b*) $E = -0.6$, (*c*) $E = 0.0$ and (*d*) $E = 200$. For the marked orbits (arrow) refer to figure 3 and figure 5.

strates this fact. The fixed points inside these regions are related to the two normal modes of the integrable system, i.e. where $a_- = 0$. The orbits in the upper and lower islands correspond, respectively, to motions along the $q_1 = q_2$ line and $q_1 = -q_2$ line.

We now want to present some observations supporting our claim that the system is integrable in some of the islands shown in figure 2. To do this we examine two of them in more detail. We stress the fact that the trajectories of the particles corresponding to this island cross the coordinate $q_i = c$, i.e. they really feel the non-linearity.

Figure 3(a) shows the island marked in figure 2(a). Because chaotic regions are usually found most easily in the vicinity of hyperbolic fixed points, we made Poincaré plots of the neighbourhood of one of them. These Poincaré sections are presented in figure 3 on different length scales. Although the magnification in figure 3(d) compared to figure 3(a) is more than 10^3, no chaos is seen. All trajectories behave in an absolutely regular manner up to a scale where noise in the numerical integration prevents further magnification. Thus it seems as if the system is integrable in this region of phase space. To check whether this kind of test is sensitive enough to detect chaos in the vicinity of hyperbolic fixed points, we performed analogous calculations for the Hamiltonian (1) but with a smooth potential (curve b in figure 1) instead of the patched one-particle potential. The smooth potential, described by a polynomial of degree 6, was chosen to approximate the former one in the most relevant region (i.e. $4 \leq x \leq 7.5$) as well as possible (cf figure 1). For low energies Poincaré plots of both potentials are qualitatively similar, showing the comparability of the two systems. Using the smooth potential and an energy comparable to that used in figure 3, we investigated one of the correspond-

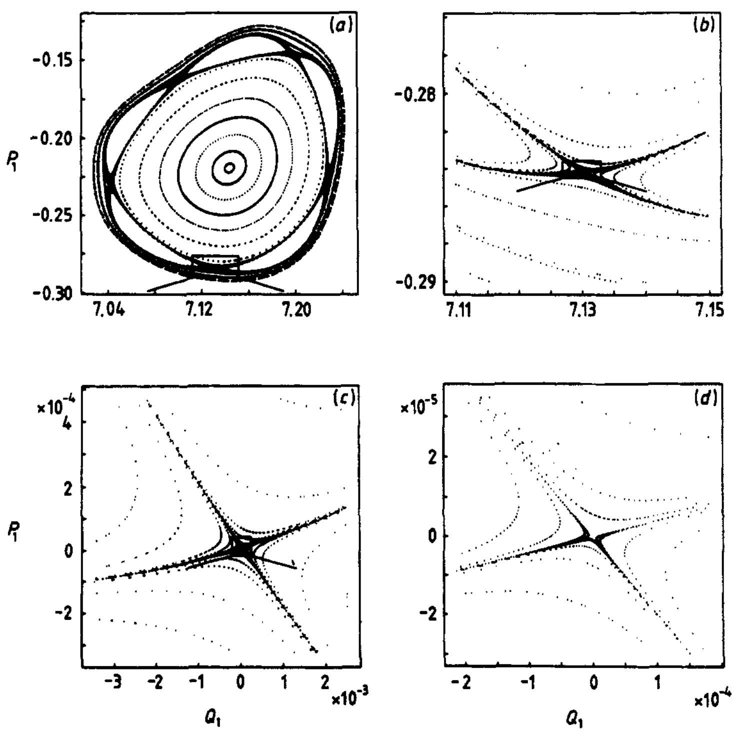

Figure 3. Successive magnifications of the island marked in figure 2(a). Note that the zero point of the coordinate system has been shifted in (c) and (d).

ing islands and performed an analysis analogous to that in figure 3. The resulting Poincaré plot for $q_2 - 5.6 = 0$ is shown in figure 4. The existence of chaotic motion near the hyperbolic fixed points is clearly visible even on this modestly magnified scale. Note that to compute these orbits we have used a fifth-order Gear algorithm, forcing the integration scheme to have (for smooth potentials!) a similar accuracy as the root-finding procedure for the piecewise parabolic potential. So the two algorithms should be comparable in accuracy. Unfortunately it is not advisable to use the Gear method for the original, non-differentiable potential because this algorithm requires the boundedness of $\dddot{q}(t)$ in order to yield accurate results. Nevertheless, we are convinced that the integrable islands in figure 3 and the chaotic behaviour in figure 4 are geniune. Thus, the unusual behaviour found in model (1) seems to be related to its piecewise harmonicity.

Another supporting fact for integrability in certain islands is the evidence for the existence of periodic orbits giving rise to lines of fixed points under some iterate of the Poincaré map. In figure 5 we show a region which is part of the island marked in figure 2(*b*). Within that island we have chosen a certain number of initial points arranged equidistantly and parallel to the p_1 axis and have calculated their higher-order iterates $(q_1^{(22n)}, p_1^{(22n)})$ (the orbits have winding number near $\frac{1}{22}$). This was done for five different,*arbitrarily* chosen sets of initial points. The results are shown in figure 5(*a*) for $n = 0,1,\ldots,6$, where the initial points as well as their higher iterates of each set are connected by lines for better visualisation. Surprisingly, for each set these lines intersect in exactly one point which must be a fixed point of the Poincaré map. This

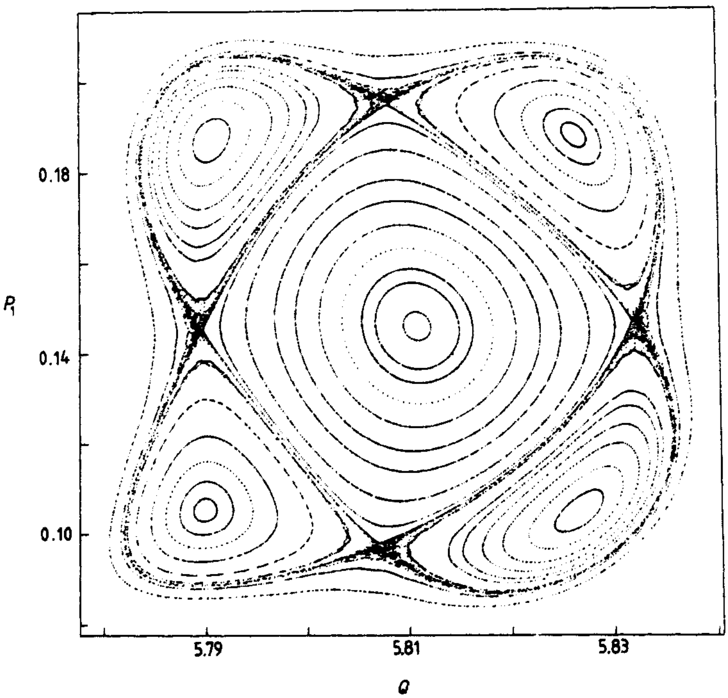

Figure 4. Magnified view of an island in the Poincaré plot of the modified potential (curve b in figure 1). The energy and the island correspond approximately to that in figure 3.

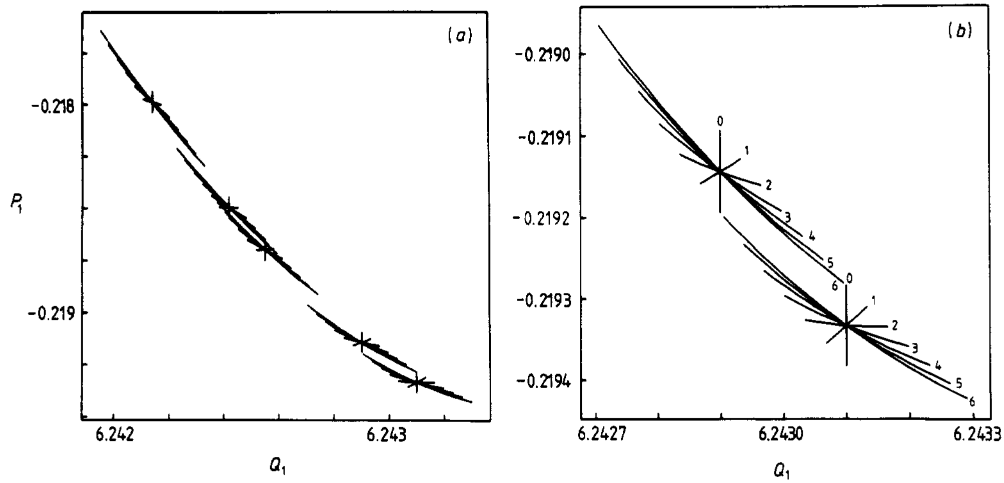

Figure 5. Magnified view of the island marked in figure 2(*b*) exhibiting the existence of fixed points with arbitrarily chosen q_1. Figure 5(*a*) shows five sets of initial configurations (vertical lines) having different values for q_1 and its higher-order iterates $(q^{(22n)}, p^{(22n)})$. In figure 5(*b*) the two lowest-lying sets of figure 5(*a*) are represented with higher resolution. The integers denote the order *n* of the higher-order iterates.

behaviour is demonstrated in figure 5(*b*) on a smaller scale. Shifting the vertical lines in figure 5(*a*) parallel to the p_1 axis, we found a finite interval for $p_1^{(0)}$ with no other fixed point of order around 22 except those in figure 5(*a*). Furthermore, each of the five fixed points belongs to five *different* periodic orbits. Because *each* of these orbits is of the *same* period 22 it is reasonable to conclude that they lie on a line, i.e. a fixed-point line as it occurs for an integrable system with rational winding number. Moving with the initial point $(q_1^{(0)}, p_1^{(0)})$ towards the centre of the island, the winding number changes and finally one hits another fixed-point line. Remembering the generic behaviour of non-integrable Hamiltonian systems with chains of alternating elliptic and hyperbolic fixed points, this is an unusual behaviour.

In this letter we have presented numerical evidence for two unusual properties of Hamiltonian systems. Firstly, we have found the existence of KAM tori although the Hamiltonian is not everywhere differentiable. This shows that the assumptions in the KAM theory are too strong. Continuity seems the only condition to be necessary for the existence of tori. Whether or not invariant tori exist for systems with $M < 3$ depends on the system itself. Secondly, we have found evidence for the coexistence (on the same energy hypersurface) of simply connected regions where the motion seems to be exactly integrable, and regions in which the motion is chaotic. This evidence is supported by the existence of fixed-point lines and the absence of chaotic behaviour in the vicinity of hyperbolic fixed points. In such a situation, there must exist an additional integral which is not defined everywhere in phase space but only on these islands. We have tried to calculate the additional integral for these islands but have not succeeded. The problem one encounters is to solve the transcendental equation (2) for t_1 and t_2. The method of Whittaker, usually employed for searching such integrals, is not applicable in this case, but perhaps other methods may succeed.

We are grateful to Professor H Thomas for many stimulating discussions. This work was supported by the Swiss National Science Foundation.

References

[1] Moser J 1973 *Stable and Random Motion in Dynamical Systems* (*Annals of Mathematical Studies* 77) (Princeton, NJ: Princeton University Press)

[2] Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (*Applied Mathematical Sciences* **38**) (Berlin: Springer)

[3] Chirikov B V 1981 *Sov. J. Plasma Phys.* **4** 289

[4] Takens F 1971 *Indag. Math.* **33** 379

[5] Bullet S 1986 *Commun. Math. Phys.* **107** 241

[6] Hénon M and Wisdom J 1983 *Physica* **8D** 157

[7] Wojtkowski M 1981 *Commun. Math. Phys.* **80** 453

[8] Lehtihet H E and Miller B N 1986 *Physica* **21D** 93

[9] Katok A B 1979 *Ann. Math.* **110** 529

[10] Przytycki F 1982 *Ergod. Theor. Dyn. Syst.* **2** 439

[11] Reichert P and Schilling R 1985 *Phys. Rev. B* **32** 5731

[12] Ali M K and Somorjai R L 1980 *Physica* **1D** 383